International Journal of Theoretical Physics, Vol. 39, No. 5, 2000

Supersmooth Topoi

Hirokazu Nishimura[1]

Received November 02, 1999

It is well known that supersmooth functions are more akin to holomorphic functions than to smooth functions. The ultimate object of study in complex geometry is not merely complex manifolds, but complex spaces, in which holomorphic functions may be nilpotent and consequently invisible from a geometric viewpoint. Supergeometers have long been searching for an appropriate definition of supermanifold, for which many classical results in the theory of smooth manifolds can naturally be superized. However, they have not proceeded further in quest of a supersmooth equivalent of complex space. The principal object of this paper is to introduce the notion of supersmooth superspace into supergeometry after the classical definition of complex space in complex geometry, and then to build a good model of synthetic supergeometry after the manner of Dubuc and Taubin (1983), thereby superseding Yetter (1988). The model to be constructed is a Grothendieck topos encompassing the category of G^∞-supermanifolds and G^∞-mappings as a full subcategory.

INTRODUCTION

Supergeometry is a fascinating subject of study to both physicists and mathematicians. It enables physicists to deal with bosons and fermions on an equal footing, leading naturally to supergravity. It is the introductory and easiest part of noncommutative geometry, whose core is by no means easy to unravel. It is a good gymnasium where novices to noncommutative geometry can raise and polish their intuitions. The central object of study in supergeometry has been no doubt *supersmooth supermanifolds*, but the exact definition of supersmooth supermanifold varies from one author to another. Furthermore, how many generators the Grassmann algebra at issue should have is disputable at best. From a physical standpoint, the convenient and natural choice of a Grassmann algebra is one with denumerably many generators, which can naturally be rendered a Banach algebra so that supersmooth

[1] Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan.

0020-7748/00/0500-1221$18.00/0 © 2000 Plenum Publishing Corporation

FIRST PAGE PROOFS

supermanifolds are Banach manifolds (Rogers, 1980), which determines odd derivatives without ambiguity, and what is more important, which leads us unquestionably to G^∞-supermanifolds as our supersmooth supermanifolds. Our choice of G^∞-functions as supersmooth functions is physically relevant, since they correspond directly to superfields, while H^∞-functions do not. As far as Grassmann algebras with infinitely many generators are concerned, the category of G^∞-supermanifolds and G^∞-mappings conforms to Rothstein's (1986) axiomatics so that we do not need to bother ourselves with such a generalization of G^∞-supermanifolds as G-supermanifolds of Bartocci *et al.* (1987, 1989) or such a modification of G^∞-functions as GH^∞-functions of Rogers (1986).

It was Grothendieck who renewed algebraic geometry in the middle of the 20th century by employing category theory (e.g., representable functors) systematically and insisting on the significance of nilpotent elements in rings. His approach culminated in what is now called scheme theory and constitutes the core of modern algebraic geometry for any neophyte to digest. Lawvere, inspired by Grothendieck's scheme theory, struggled in the 1960s to revive the then moribund idea of nilpotent infinitesimals in differential geometry by exploiting his favorite machinery of category theory. While the significance of the former approach is now well recognized and appreciated in the contemporary mathematical community, the latter approach, referred to as *synthetic differential geometry*, remains in the doldrums and has not attracted the attention and momentum that it deserves. The so-called tensor analysis on infinitesimal entities (e.g., vector fields) in orthodox differential geometry is often stodgy and factitious, concealing the truly infinitesimal nature of infinitesimal considerations under a topsy-turvy of lengthy calculations in a dull drone. Synthetic differential geometry enables us to endow differential geometry with an infinitesimal horizon relatively independent of local and global ones. For good textbooks on synthetic differential geometry the reader is referred to Lavendhomme (1996) (devoted mainly to a consistently axiomatic presentation of synthetic differential geometry) and Moerdijk and Reyes (1991) (devoted to model theory of synthetic differential geometry) as well as Kock's 1981b bible of the field.

Nishimura (1998, 1999, 2000) has developed *synthetic supergeometry* up to superconnections from an axiomatic standpoint, while Yetter (1998) has made a fresh start in model theory of synthetic supergeometry, committing himself to H^∞-functions and graded manifolds. It is well known that graded manifolds are essentially tantamount to DeWitt supermanifolds, whatever definition of supersmooth function we adopt (H^∞, G^∞, or GH^∞), for which the reader is referred to Batchelor (1980) and Bartocci *et al.* (1991, Chapter V, §4). It is also well known that a DeWitt supermanifold or a graded manifold is virtually equivalent to the exterior bundle of a vector bundle over a smooth

FIRST PAGE PROOFS

manifold, as was demonstrated by Batchelor (1979). This is why Yetter (1988) was able to transfer so smoothly from the smooth paradigm to the supersmooth or rather graded paradigm. Since not all supersmooth supermanifolds are DeWitt supermanifolds and, as we have claimed, H^∞-functions are not physically relevant, we should say that Yetter's (1988) model theory of synthetic supergeometry is somewhat too restrictive in our present paradigm of supergeometry. The leitmotif of this paper is to give a good model of synthetic supergeometry, possibly to be called a *supersmooth topos*, which is a Grothendieck topos containing the category of G^∞-supermanifolds and G^∞-mappings as a full subcategory.

It is well known that supersmooth functions are more akin to holomorphic functions of complex analysis than to smooth functions of real analysis. Therefore it is natural to build our model theory of synthetic supergeometry not directly after the standard manner of Moerdijk and Reyes (1991), but after the manner of Dubuc and Taubin (1983). This consequently forces us to coin a supersmooth equivalent of the classical notion of complex space in complex geometry (Grauert and Remmert, 1984), which is much more general than the notion of supersmooth supermanifold. *Supersmooth superspaces* to be introduced in Section 3 are the desired equivalent, and will play the same role in our model theory of synthetic supergeometry as finitely generated C^∞-rings have played in the standard model theory of synthetic differential geometry. Once supersmooth superspaces are introduced, we can follow the standard route to build the desired Grothendieck topos as an appropriate model of synthetic supergeometry, which we will do in Section 4. Sections 1 and 2 are devoted respectively to a review of supersmooth functions and a treatment of *supersmooth superrings* after Dubuc and Taubin's (1983) good exposition of analytic rings.

1. SUPERSMOOTH FUNCTIONS

We denote by B_∞ the Grassmann algebra over denumerable generators in the sense of Rogers (1980, p. 1353), which is naturally to be seen as a Banach algebra. The even part of B_∞ is denoted by $B_\infty^{\mathbf{0}}$, while its odd part is denoted by $B_\infty^{\mathbf{1}}$. We denote $(B_\infty^{\mathbf{0}})^m \times (B_\infty^{\mathbf{1}})^n$ by $B_\infty^{m,n}$ ($m, n \geq 0$). We note that $B_\infty^{0,0}$ consists of a single point. A smooth function f from an open subset U of $B_\infty^{m,n}$ to B_∞ is said to be $\mathbf{G}^\infty$ or *supersmooth* if its Fréchet derivative $\mathbf{D}f$ is $B_\infty^{\mathbf{0}}$-linear at each point of U. For the equivalence of this friendly definition of G^∞ to Rogers' (1980) original one, the reader is referred to Jadczyk and Pilch (1981, §5). Given an open subset U of $B_\infty^{m,n}$, we denote by $\mathcal{G}^\infty(U)$ the sheaf of germs of supersmooth functions into B_∞. We denote by $\mathbf{O}$ the category of open subsets of $B_\infty^{m,n}$'s as objects and supersmooth functions as morphisms.

FIRST PAGE PROOFS

Hadamard's lemma and the implicit function theorem for G^∞, which are to be presented in the following, are fundamental in the development of our theory of supersmooth topoi.

Lemma 1.1. Let U be a convex open subset U of $B_\infty^{1,0}$ or $B_\infty^{0,1}$ and V an open subset of $B_\infty^{m,n}$. Let f: $U \times V \to B_\infty$ be a supersmooth function. Then there exists a supersmooth. function g: $U \times U \times V \to B_\infty$ such that

$$(1.1) \quad f(y, z) - f(x, z) = (y - x)\, g(x, y, z)$$

for any $(x, y, z) \in U \times U \times V$.

Proof. Take $g(x, y, z) = \int_0^1 \mathbf{D}_1 f(x + t(y - x), z)\, dt$, where $\mathbf{D}_1 f$ denotes the partial derivative of f with respect to the first component. ■

Theorem 1.2. Let $U \subseteq B_\infty^{m_1,n_1}$ and $V \subseteq B_\infty^{m_2,n_2}$ be open subsets, and f: $U \times V \to B_\infty^{m_3,n_3}$ a supersmooth function such that $f(x_0, y_0) = 0$ at some $(x_0, y_0) \in U \times V$. Let us suppose that the second partial derivative $\mathbf{D}_2 f(x_0, y_0)$ of f at (x_0, y_0) is invertible. Then there exists an open neighborhood W of x_0 in $B_\infty^{m_1,n_1}$ and a supersmooth function g: $W \to V$ such that $f(x, g(x)) = 0$ for all $x \in W$.

Proof. See, e.g., Theorem 2.8 of Cianci (1990). ■

2. SUPERSMOOTH SUPERRINGS

Let **E** be a finitely complete category. Given two functors F and G from **0** to **E**, a natural transformation π: $F \to G$ is said to be *local* if for all open inclusions $U \subset V$ in **O** the square

$$\begin{array}{ccc} F(U) & \rightarrow & F(V) \\ {\scriptstyle \pi_U}\downarrow & & \downarrow{\scriptstyle \pi_V} \\ G(U) & \rightarrow & G(V) \end{array}$$

is a pullback in **E**.

A *supersmooth superring* in a finitely complete category **E** is a functor F: **O** $\to$ **E** preserving transversal pullbacks and terminal objects. Morphisms of supersmooth superrings are simply natural transformations. It is easy to see that $F(B_\infty^{1,1})$ is a B_∞-superalgebra whose even and odd parts are respectively $F(B_\infty^{1,0})$ and $F(B_\infty^{0,1})$.

As in Theorem 1.10 of Dubuc and Taubin (1983), the implicit function theorem for supersmooth functions discussed in Theorem 1.2 gives the following fundamental result on supersmooth superrings:

Theorem 2.1. Let **E** be a finitely comlete category and F: **O** $\to$ **E** a functor preserving finite products and terminal objects. If there is a local natural transformation π: $F \to G$ with G a supersmooth superring in **E** preserving open coverings, then F is a supersmooth superring in **E** preserving open coverings.

Since the category **Sets** of sets and functions is finitely complete, we will consider supersmooth superrings in **Sets**. Obviously the identity functor of **O** into **Sets** is a supersmooth superring in **Sets**, which is denoted by B_∞ in abuse of language.

As in Theorem 1.18 of Dubuc and Taubin (1983), Lemma 1.1 and Theorem 2.1 give

Theorem 2.2. Let π: $G \to B_\infty$ be a local morphism of supersmooth superrings in **Sets**. Let I be a superideal of the B_∞-superalgebra $G(B_\infty^{1,1})$. Then there exists a unique supersmooth superring, denoted by G/I, together with a unique morphism ν: $G \to G/I$ of supersmooth superrings and a unique morphism π': $\to G/I \to B_\infty$ of supersmooth superrings such that $(G/I)(B_\infty^{1,1}) = G(B_\infty^{1,1})/I$, $\nu_{B_\infty^{1,1}}$ is equal to the canonical morphism $G(B_\infty^{1,1}) \to G(B_\infty^{1,1})/I$ as B_∞-superalgebras, and $\pi = \pi' \circ \nu$.

3. SUPERSMOOTH SUPERSPACES

Given a topological space X, we denote by $\mathbf{S}_X$ the category of sheaves of sets on X, which is surely finitely complete. We denote by **Top** the category of topological spaces and continuous functions. The assignment of the object **Top** $(\cdot, U)$ in $\mathbf{S}_X$ to each object U in **O** determines a functor from **O** to $\mathbf{S}_X$, which is surely a supersmooth superring in $\mathbf{S}_X$ and which is denoted by $\mathcal{B}_{\infty,X}$. An ordered pair $(X, \mathcal{F}_X)$ of a topological space X and a supersmooth superring $\mathcal{F}_X$ in $\mathbf{S}_X$ is called a *supersmoothly superringed superspace* if there exists a local morphism π: $\mathcal{F}_X \to \mathcal{B}_{\infty,X}$ of supersmooth superrings. The notion of a morphism of supersmoothly superringed superspaces can be defined after the manner of ringed spaces, and the resulting category of supersmoothly superringed superspaces is denoted by $\overline{\mathbf{SupS}}$. Given an object U in **O**, the assignment of the object $\mathbf{O}(\cdot, V)$ in $\mathbf{S}_U$ to each object V in **O** determines a functor from **O** to $\mathbf{S}_U$, which is surely a supersmooth superring in $\mathbf{S}_U$ furnished with the canonical local morphism of supersmooth superrings into $\mathcal{B}_{\infty,U}$ and which is denoted by $\mathcal{G}_U^\infty$. Thus the ordered pair $(U, \mathcal{G}_U^\infty)$ is a supersmoothly superringed superspace, and the assignment of the supersmoothly superringed superspace $(U, \mathcal{G}_U^\infty)$ to each object U in **O** determines a functor from **O** to $\overline{\mathbf{SupS}}$ to be denoted by $\mathbf{i}_{\mathbf{O},\overline{\mathbf{SupS}}}$. The underlying topological space X and the structure sheaf $\mathcal{F}_X$ of a supersmoothly superringed superspace $(X, \mathcal{F}_X)$ is denoted by $\Pi_1((X, \mathcal{F}_X))$ and $\Pi_2((X, \mathcal{F}_X))$.

By the same token as Dubuc and Taubin's (1983). Theorem 2.8 we have:

FIRST PAGE PROOFS

Theorem 3.1. For any object $(X, \mathcal{F}_X)$ in $\overline{\mathbf{SupS}}$ we have the following commutative diagram, where Γ denotes the global sections functor:

$$\begin{array}{ccc} \mathbf{O} & \xrightarrow{\mathbf{i}_{\mathbf{O},\overline{\mathbf{SupS}}}} & \overline{\mathbf{SupS}} \\ {\scriptstyle \mathcal{F}_X}\downarrow & & \downarrow{\scriptstyle \overline{\mathbf{SupS}}((X,\,\mathcal{F}_X),\,\cdot)} \\ \boldsymbol{S}_X & \xrightarrow{\Gamma} & \mathbf{Sets} \end{array}$$

By the same token as Dubuc and Taubin's (1983), Theorem 2.10, we have:

Theorem 3.2. Given an object U in $\mathbf{O}$ and an arbitrary sheaf $\mathcal{Y}$ of superideals in the sheaf $\mathcal{G}^\infty(U)$ of superrings, we define E to be the support Supp $(\mathcal{G}^\infty(U)/\mathcal{Y})$ of the quotient sheaf $\mathcal{G}^\infty(U)/\mathcal{Y}$. Then there exists a unique supersmooth superring $\mathcal{G}_E^\infty$ in $\mathbf{S}_E$ together with a local morphism l_E: $\mathcal{G}_E^\infty \to \mathcal{B}_{\infty,E}$ of supersmooth superrings such that the superring $\mathcal{G}_E^\infty(\mathbf{B}_\infty^{1,1})$ in $\mathbf{S}_E$ is equal to the restriction of the sheaf $\mathcal{G}^\infty(U)/\mathcal{Y}$ to E. Thus the pair $(E, \mathcal{G}_E^\infty)$ is naturally a supersmoothly superringed superspace.

In the above theorem, if the sheaf $\mathcal{Y}$ of superideals is of finite type, then the resulting supersmoothly superringed superspace $(E, \mathcal{G}_E^\infty)$ is called a *supersmooth model superspace*. A supersmoothly superringed superspace $(X, \mathcal{F}_X)$ is called a *supersmooth superspace* if the topological space X is Hausdorff and every point x of X has an open neighborhood U such that the restriction of the supersmoothly superringed superspace $(X, \mathcal{F}_X)$ to U is isomorphic to a supersmooth model superspace. We denote by **SupS** the full subcategory of $\overline{\mathbf{SupS}}$ whose objects are all supersmooth superspaces. The category of supersmooth supermanifolds and supersmooth mappings in the sense of Rogers (1980) is denoted by **SupM**.

It is straightforward to show that:

Proposition 3.3. The canonical embedding $\mathbf{i}_{\mathbf{SupM},\,\mathbf{SupS}}$: $\mathbf{SupM} \to \mathbf{SupS}$ is full and faithful, and preserves transversal pullbacks and terminal objects.

Proof. By Bartocci *et al.* (1991, Chapter II, Corollary 4.3). ■

By such a familiar token of algebraic geometry and complex geometry as seen in Hartshorne (1977, Chapter II, Theorem 3.3) and Grauert and Remmert (1984, Chap. 1, §3. Theorem 4) we have

Theorem 3.4. The category **SupS** is finitely complete.

4. SUPERSMOOTH TOPOI

We will first show that the topos $\mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$ has many features of a good model of synthetic supergeometry, and then will indicate in brief how to modify it so as to get a better model.

FIRST PAGE PROOFS

The following proposition is an instance of the well-known fundamental result on Yoneda embeddings (cf. (MacLane, 1971, Chapter II, §2).

Proposition 4.1. The Yoneda embedding $\mathbf{y}_{\mathbf{SupS}}$: $\mathbf{SupS} \to \mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$ is full and faithful.

We denote the composition $\mathbf{y}_{\mathbf{SupS}} \circ \mathbf{i}_{\mathbf{SupM},\,\mathbf{SupS}}$ by $\mathbf{y}_{\mathbf{SupM}}$.

Corollary 4.2. The embedding $\mathbf{y}_{\mathbf{SupM}}$: $\mathbf{SupM} \to \mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$ is full and faithful.

Proof. This follows directly from Propositions 3.3 and 4.1. ■

The following proposition is an instance of a well-known fact on Yoneda embeddings (cf. Schubert, 1972, Theorem 10.2.5).

Proposition 4.3. The Yoneda embedding $\mathbf{y}_{\mathbf{SupS}}$: $\mathbf{SupS} \to \mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$ preserves limits.

Corollary 4.4. The embedding $\mathbf{y}_{\mathbf{SupM}}$: $\mathbf{SupM} \to \mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$ preserves transversal pullbacks and terminal objects.

Proof. This follows directly from Propositions 3.3 and 4.3. ■

In our synthetic supergeometry (Nishimura, 1998, 1999, n.d.) the set of superreal numbers is denoted by $\mathbb{R}$ and is required to abide by an appropriate superization of the general Kock axiom. The assignment of the set $\Gamma(\mathcal{F}_X(B_\infty^{1,1}))$ of global sections of the sheaf $\mathcal{F}_X(B_\infty^{1,1})$ to each supersmooth superspace $(X, \mathcal{F}_X)$ naturally gives rise to a contravariant functor from **Sups** to **Sets**, which shall be the interpretation of $\mathbb{R}$ in the topos $\mathbf{SetS}^{\mathbf{SupS}^{\mathrm{op}}}$. We will loosely use the same symbol for both such an entity of synthetic supergeometry as $\mathbb{R}$ and its interpretation in such an appropriate topos as $\mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$. It is easy to show that:

Proposition 4.5. $\mathbb{R}$ is a superring in the topos $\mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$.

Proposition 4.6. The contravariant functor $\mathbb{R}$ from the category **SupS** to the category **Sets** is representable with its representing object $(B_\infty^{1,1}, \mathcal{G}^\infty_{B_\infty^{1,1}})$.

Proof. This follows readily from Theorem 3.2, just as Corollary 2.9 of Dubuc and Taubin (1983) follows from their Theorem 2.8. ■

Theorem 4.7. The contravariant functor $\mathbb{R}^{\mathbb{R}}$ from the category **SupS** to the category **Sets** is naturally isomorphic to the contravariant functor assigning $\Gamma(\Pi_2((X, \mathcal{F}_X) \times (B_\infty^{1,1}, \mathcal{G}^\infty_{B_\infty^{1,1}}))(B_\infty^{1,1}))$ to each supersmooth superspace $(X, \mathcal{F}_X)$.

Proof. We have that

FIRST PAGE PROOFS

$$
\begin{aligned}
(4.1)\quad & \mathbb{R}^{\mathbb{R}}((X, \mathcal{F}_X)) \\
&= \mathbf{Sets}^{\mathbf{SupS}^{op}}(\mathbf{SupS}(\cdot, (X, \mathcal{F}_X)) \times \mathbb{R}, \mathbb{R}) \\
&\quad \text{[Formula (5) of MacLane and Moerdijk (1992,} \\
&\quad \text{Chapter I, §6)]} \\
&= \mathbf{Sets}^{\mathbf{SupS}^{op}}(\mathbf{SupS}(\cdot, (X, \mathcal{F}_X)) \times \mathbf{SupS}(\cdot, (B_\infty^{1,1}, \mathcal{G}^\infty_{B_\infty^{1,1}})), \\
&\quad \mathbf{SupS}(\cdot, (B_\infty^{1,1}, \mathcal{G}^\infty_{B_\infty^{1,1}}))) \quad \text{[Proposition 4.6]} \\
&= \mathbf{Sets}^{\mathbf{SupS}^{op}}(\mathbf{SupS}(\cdot, (X, \mathcal{F}_X) \times (B_\infty^{1,1}, \mathcal{G}^\infty_{B_\infty^{1,1}})), \\
&\quad \mathbf{SupS}(\cdot, (B_\infty^{1,1}, \mathcal{G}^\infty_{B_\infty^{1,1}}))) \\
&= \mathbf{SupS}((X, \mathcal{F}_X) \times (B_\infty^{1,1}, \mathcal{G}^\infty_{B_\infty^{1,1}}), (B_\infty^{1,1}, \mathcal{G}^\infty_{B_\infty^{1,1}})) \\
&\quad \text{[Yoneda lemma]} \\
&= \Gamma(\Pi_2((X, \mathcal{F}_X) \times (B_\infty^{1,1}, \mathcal{G}^\infty_{B_\infty^{1,1}}))(B_\infty^{1,1})) \\
&\quad \text{[Proposition 4.6]}
\end{aligned}
$$

which establishes the desired theorem.

Now we will discuss the interpretation of the spectrum $\mathrm{Spec}_{\mathbb{R}}(\mathfrak{M})$ of a Weil superalgebra $\mathfrak{M}$ in $\mathbb{R}$ within the topos $\mathbf{Sets}^{\mathbf{SupS}^{op}}$. It is easy to see that:

Proposition 4.8. The interpretation of $\mathfrak{M}$ in the topos $\mathbf{Sets}^{\mathbf{SupS}^{op}}$ is the contravariant functor assigning $\Gamma(\mathcal{F}_X(B_\infty^{1,1})) \times \mathfrak{M}((B_\infty^{0,0}, \mathcal{G}^\infty_{B_\infty^{0,0}}))$ to each supersmooth superspace $(X, \mathcal{F}_X)$.

It is easy to show that:

Proposition 4.9. The contravariant functor $\mathrm{Spec}_{\mathbb{R}}(\mathfrak{M})$ from the category **SupS** to the category **Sets** is representable with its representing object $(B_\infty^{0,0}, \mathcal{M}((B_\infty^{0,0}, \mathcal{G}^\infty_{B_\infty^{0,0}})))$.

Theorem 4.10. The contravariant functor $\mathbb{R}^{\mathrm{Spec}_{\mathbb{R}}(\mathfrak{M})}$ from **SupS** to **Sets** is naturally isomorphic to the contravariant functor $\mathfrak{M}$.

Proof. For each supersmooth superspace $(X, \mathcal{F}_X)$ we have that

$$
\begin{aligned}
(4.2)\quad & \mathbb{R}^{\mathrm{Spec}_{\mathbb{R}}(\mathfrak{M})}((X, \mathcal{F}_X)) \\
&= \mathbf{Sets}^{\mathbf{SupS}^{op}}(\mathbf{SupS}(\cdot, (X, \mathcal{F}_X)) \times \mathrm{Spec}_{\mathbb{R}}(\mathfrak{M}), \mathbb{R}) \\
&\quad \text{[Formula (5) of Maclane and Moerdijk (1992,} \\
&\quad \text{Chapter I, §6)]}
\end{aligned}
$$

FIRST PAGE PROOFS

$$
\begin{aligned}
&= \mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}(\mathbf{SupS}(\cdot, (X, \mathcal{F}_X)) \\
&\quad \times \mathbf{SupS}(\cdot, (B_\infty^{0,0}, \mathfrak{M}((B_\infty^{0,0}, \mathcal{G}^\infty_{B_\infty^{0,0}})))), \\
&\quad \mathbf{SupS}(\cdot, (B_\infty^{1,1}\mathcal{G}^\infty_{B_\infty^{1,1}}))) \\
&\quad \text{[Propositions 4.6 and 4.9]} \\
&= \mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}(\mathbf{SupS}(\cdot, (X, \mathcal{F}_X) \times (B_\infty^{0,0}, \mathfrak{M}((B_\infty^{0,0}, \mathcal{G}^\infty_{B_\infty^{0,0}})))) \\
&\quad \mathbf{SupS}(\cdot, (B_\infty^{1,1}, \mathcal{G}^\infty_{B_\infty^{1,1}}))) \\
&= \mathbf{SupS}((X, \mathcal{G}_X) \times (B_\infty^{0,0}, \mathfrak{M}((B_\infty^{0,0}, \mathcal{G}^\infty_{B_\infty^{0,0}}))) \\
&\quad (B_\infty^{1,1}, \mathcal{G}^\infty_{B_\infty^{1,1}})) \qquad \text{[Yoneda lemma]} \\
&= \Gamma(\Pi_2((X, \mathcal{F}_X) \times (B_\infty^{0,0}, \mathfrak{M}((B_\infty^{0,0}, \mathcal{G}^\infty_{B_\infty^{0,0}}))))(B_\infty^{1,1})) \\
&\quad \text{[Proposition 4.6]} \\
&= \Gamma(\mathcal{F}_X(B_\infty^{1,1})) \times \mathfrak{M}((B_\infty^{0,0}, \mathcal{G}^\infty_{B_\infty^{0,0}})) \\
&= \mathfrak{M}((X, \mathcal{F}_X)) \qquad \text{[Proposition 4.8]}
\end{aligned}
$$

which establishes the desired theorem. ■

The embedding $\mathbf{y}_{\mathrm{SupS}}$: $\mathbf{SupS} \to \mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$ does not preserve open covers simply because not all the functors in $\mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$ believe this. By cutting down the universe $\mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$ to those functors which believe that all open covers in **SupS** are covers in $\mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$, we get a better universe for synthetic supergeometry to be called the *supersmooth Zariski topos*[2] and to be denoted by $\mathbf{Z}_{\mathbf{SupS}}$. Formally the topos $\mathbf{Z}_{\mathbf{SupS}}$ is obtained by considering open covers as covering families of a Grothendieck topology on **SupS** and then considering all the sheaves with respect to this Grothendieck topology.

It is easy to show that:

Proposition 4.11. The Grothendieck topology is subcanonical. In other words, the Yoneda embedding $\mathbf{y}_{\mathbf{SupS}}$: $\mathbf{SupS} \to \mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$ factors through $\mathbf{Z}_{\mathbf{SupS}} \to \mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$

This guarantees that statements about $\mathbf{Z}_{\mathbf{SupS}}$ correspond directly to statements about supersmooth superspaces. Now it is not difficult to show in a standard way that the topos $\mathbf{Z}_{\mathbf{SupS}}$ not only enjoys all the properties of its preceding topos $\mathbf{Sets}^{\mathbf{SupS}^{\mathrm{op}}}$ discussed so far (surely with due modifications),

[2] We now feel that what is called the *supersmooth Zariski topos* in the paper should be called the *supersmooth Dubuc topos*.

FIRST PAGE PROOFS

but also preserves open covers, the details of which are safely left to the reader (cf. MacLane and Moerdijk, 1991. Chapters III and VI).

REFERENCES

Bartocci, C., and Bruzzo, U. (1987). Some remarks on the differential-geometric approach to supermanifolds. *Journal of Geometry and Physics*, **4**, 391–404.

Bartocci, C., Bruzzo, U., and Hernández-Ruipérez, D. (1989). A remark on a new category of supermanifolds, *Journal of Geometry and Physics*, **6**, 509–516.

Bartocci, C., Bruzzo, U., and Hernández-Ruipérez, D. (1991). *The Geometry of Supermanifolds*, Kluwer, Dordrecht.

Batchelor, M. (1979). The structure of supermanifolds, *Transactions of the American Mathematical Society*, **253**, 329–338.

Batchelor, M. (1980). Two approaches to supermanifolds, *Transactions of the American Mathematical Society*, **258**, 257–270.

Berezin, F. A. (1987). *Introduction to Superanalysis*, Reidel, Dordrecht.

Borceux, F. (1994). *Handbook of Categorical Algebra*, Cambridge University Press, Cambridge.

Boyer, C. P., and Gitler, S. (1984). The theory of G^∞-supermanifolds, *Transactions of the American Mathematical Society*, **285**, 241–267.

Bunge, M., and Dubuc, E. J. (1986). Archimedian local C^∞-rings and models of synthetic differential geometry, *Cahiers Topologie Géométrie Différentielle Catégoriques*, **27**, 3–22.

Cianci, R. (1990). *Introduction to Supermanifolds*, Bibliopolis, Napoles, Italy.

Demazure, M., and Gabriel, P. (1980). *Introduction to Algebraic Groups*, North-Holland, Amsterdam.

DeWitt, B. (1984). *Supermanifolds*, Cambridge University Press, Cambridge.

Dieudonné, J. (1960). *Foundations of Modern Analysis*, Academic Press, New York.

Dubuc, E. J. (1979). Sur les modèles de la géométrie differentielle synthétique, *Cahiers Topologie Géométrie Differentielle Catégoriques*, **20**, 234–279.

Dubuc, E. J. (1981). C^∞-schemes, *American Journal of Mathematics*, **103**, 683–690.

Dubuc, E. J. (1986). Logical opens and real numbers in topoi, *Journal of Pure and Applied Algebra*, **43**, 129–143.

Dubuc, E. J., and Taubin, G. (1983). Analytic rings, *Cahiers Topologie Géométrie Différentielle Catégoriques*, **24**, 225–265.

Eisenbud, D. (1995). *Commutative Algebra with a View toward Algebraic Geometry*, Springer-Verlag, New York.

Gawedzki, K. (1977). Supersymmetries—mathematics of supergeometry, *Annales de l'Institute Henri Poincaré, Section A.* **27**, 335–366.

Grauert, H., and Remmert, R. (1984). *Coherent Analytic Sheaves*, Springer-Verlag, Berlin.

Gunning, R. C., and Rossi, H. (1965). *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, New Jersey.

Hakim, M. (1972). *Topos Annelés et Schémas Relatifs*, Springer-Verlag, Berlin.

Hartshorne, R. (1977). *Algebraic Geometry*, Springer-Verlag, New York.

Horikawa, E. (1990). *An Introduction to Complex Algebraic Geometry* [in Japanese], Iwanami, Tokyo.

Jadczyk, A., and Pilch, K. (1981). Superspaces and supersymmetries, *Communications in Mathematical Physics*, **78**, 373–390.

Kock, A. (1981a). Properties of well-adapted models for synthetic differential geometry, *Journal of Pure and Applied Algebra*, **20**, 55–70.

Kock, A. (1981b). *Synthetic Differential Geometry*, Cambridge University Press, Cambridge.

FIRST PAGE PROOFS

Kostant, B. (1977). Graded manifolds, graded Lie theory and prequantization, In *Lecture Notes in Mathematics*, No. 570. Springer-Verlag, Berlin, pp. 177–306.

Lang, S. (1995). *Differential and Riemannian Manifolds*, Springer-Verlag, New York.

Lavendhomme, R. (1996). *Basic Concepts of Synthetic Differential Geometry*, Kluwer, Dordrecht.

Leites, D. A. (1980). Introduction to the theory of supermanifolds, *Russian Mathematical Surveys*, **35**, 1–64.

MacLane, S. (1971). *Categories for the Working Mathematician*, Springer-Verlag, New York.

MacLane, S., and Moerdijk, I. (1992). *Sheaves in Geometry and Logic*, Springer-Verlag, New York.

Manin, Y. I. (1988). *Gauge Field Theory and Complex Geometry*, Springer-Verlag, Heidelberg.

Moerdijk, I. (1987). On the embedding of manifolds into the smooth Zariski topos. In *Mathematical Logic and Theoretical Computer Science*, D. W. Kuecker, E. G. K. Lopez-Escobar, and C. H. Smith, eds., Marcel Dekker, New York.

Moerdijk, I., and Reyes, G. E. (1986). Rings of smooth functions and their localizations, I, *Journal of Algebra*, **99**, 324–336.

Moerdijk, I., and Reyes, G. E. (1991). *Models for Smooth Infinitesimal Analysis*, Springer-Verlag, New York.

Moerdijk, I., Ngo van Quê, and Reyes, G. E. (1987). Rings of smooth functions and their localizations, II, In *Mathematical Logic and Theoretical Computer Science*, D. W. Kuecker, E. G. K. Lopez-Escobar, and C. H. Smith, eds., Marcel Dekker, New York.

Năstăsescu, C., and van Oystaeyen, F. (1982). *Graded Ring Theory*, North-Holland, Amsterdam.

Nishimura, H. (1998). Synthetic differential supergeometry, *International Journal of Theoretical Physics*, **37**, 2803–2822.

Nishimura, H. (1999). Differential forms in synthetic differential supergeometry, *International Journal of Theoretical Physics*, **38**, 653–663.

Nishimura, H. (2000). Synthetic theory of superconnections, *International Journal of Theoretical Physics*, **39**, 297–320.

Rogers, A. (1980). A global theory of supermanifolds, *Journal of Mathematical Physics*, **21**, 1352–1365.

Rogers, A. (1986). Graded manifolds, supermanifolds and infinite-dimensional Grassmann algebras, *Communications in Mathematical Physics*, **105**, 375–384.

Rothstein, M. J. (1986). The axioms of supermanifolds and a new structure arising from them, *Transactions of the American Mathematical Society*, **297**, 159–180.

Schubert, H. (1972). *Categories*, Springer-Verlag, Berlin.

Yetter, D. N. (1988). Models for synthetic supergeometry, *Cahiers Topologie Géométrie Différentielle Catégoriques*, **29**, 87–108.

FIRST PAGE PROOFS